

# Icons

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## Abstract

Categorical orthodoxy has it that collections of ordinary mathematical structures such as groups, rings, or spaces, form *categories* (such as the category of groups); collections of 1-dimensional categorical structures, such as categories, monoidal categories, or categories with finite limits, form *2-categories*; and collections of 2-dimensional categorical structures, such as 2-categories or bicategories, form *3-categories*.

We describe a useful way in which to regard bicategories as objects of a 2-category. This is a bit surprising both for technical and for conceptual reasons. The 2-cells of this 2-category are the crucial new ingredient; they are the icons of the title. These can be thought of as “the oplax natural transformations whose components are identities”, but we shall also give a more elementary description.

We describe some properties of these icons, and give applications to monoidal categories, to 2-nerves of bicategories, to 2-dimensional Lawvere theories, and to bundles of bicategories.

## 1 Introduction

For any particular mathematical structure, there is a category whose objects are instances of that structure, and whose morphisms are the structure-preserving maps. For the structure of group, we have the category **Grp** of groups and group homomorphisms. Similarly, for the structure of category itself, we have the category **Cat**<sub>1</sub> of categories and functors. But in this case,

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as well as functors, there are also natural transformations between functors, and the category of categories naturally becomes a 2-category called **Cat**<sub>2</sub> or just **Cat**.

What happens when the mathematical structure in question is a 2-dimensional one, such as a bicategory? Once again, one could use the structure preserving maps to obtain a category of bicategories (although there is a certain amount of choice in what exactly is meant by “structure-preserving”), but now there are two further levels of structure, which altogether give a 3-dimensional structure called a tricategory [4] or weak 3-category. In particular, there is a tricategory called **Hom** [19, 4] whose objects are the bicategories. There is no doubt that this is a fine and excellent structure. But my goal here is to argue for the merits of a different context in which to consider bicategories. This different context will be just 2-dimensional, so that we get a 2-category of bicategories. This is not done by merely throwing away the 3-cells of **Hom**— for one thing this would not give a 2-category or even a bicategory, but even if it did it would conceptually not be the right thing to do. Rather, we introduce a new type of 2-cell, which we call an *icon*, and offer the following propaganda for these new 2-cells.

**They provide a relatively simple context in which to study bicategories.** The 3-dimensional structure **Hom** is, as mentioned above, a tricategory [4], and tricategories are rather complex structures. Although for some purposes one really needs to use this full tricategory, if one can get by with just a 2-category of bicategories life is very much simpler. For example, this allows bicategories to be treated using the techniques of 2-dimensional universal algebra [2], which is very much more developed than 3-dimensional universal algebra [17].

**They arise naturally if one thinks of 2-categories or bicategories as “many-object” monoidal categories.** One-object categories are essentially equivalent to monoids. If you think of monoids as one-object categories, then a homomorphism of monoids is just a functor. Thus the category of monoids is a full subcategory of the category **Cat**<sub>1</sub> of categories and functors; this does not use the 2-category structure **Cat** built upon **Cat**<sub>1</sub>.

Similarly, one-object bicategories are essentially equivalent to monoidal categories, and if you think of monoidal categories as one-object bicategories, then structure preserving morphisms of bicategories are the same as structure preserving morphisms of monoidal categories. But between

the structure-preserving morphisms of monoidal categories there are things called monoidal natural transformations, and these are precisely the icons (in the case of one-object bicategories).

**They facilitate the construction of 2-nerves of bicategories.** The nerve construction is of great importance in algebraic topology and elsewhere. It associates to every category a simplicial set; in fact it provides a full embedding of the category  $\mathbf{Cat}_1$  of categories and functors into the category  $[\Delta^{\text{op}}, \mathbf{Set}]$  of simplicial sets. This construction is a formal consequence of the fact that there is an inclusion of  $\Delta$  in  $\mathbf{Cat}_1$ . The 2-nerve construction associates to every bicategory a simplicial category. We can obtain this as a formal consequence of the fact that  $\Delta$  embeds in a suitably defined 2-category of bicategories, with icons as 2-cells, as was proved in [13].

**They arise in connection with 2-dimensional Lawvere theories.** Lawvere theories are used to describe one-dimensional algebraic structures, such as groups, rings, or Lie algebras. 2-dimensional Lawvere theories [16, 14] are used to describe two-dimensional algebraic structures, such as monoidal categories or categories with finite limits. Formally, a Lawvere theory is a particular type of category, and a 2-dimensional Lawvere theory is a special type of 2-category. A morphism of 2-dimensional Lawvere theories is a special type of 2-functor, and a transformation between such morphisms is a special type of icon. These transformations play a key role in a notion called *flexibility*, which distinguishes structures such as *monoidal categories* which transport along equivalences, to structures such as *strict monoidal categories*, which do not.

**They allow the use of notions internal to a 2-category, such as equivalence and fibration.** Various notions such as equivalence and fibration can be defined internally to any 2-category; in particular this can be done in the various 2-categories of bicategories defined using icons. Equivalence in these 2-categories is quite a strict notion, but it can be proved that any bicategory is equivalent, in a suitable 2-category of bicategories with icons as 2-cells, to a strict bicategory (that is, a 2-category) [13].

On the other hand a fibration in the various 2-categories of bicategories can be thought of as a sort of “bundle of bicategories” [7].

## 2 Bicategorical preliminaries

As indicated above, and as first observed in [1], there are various possible notions of morphism of bicategory. First one could consider those which literally preserve all of the structure, so that for instance, given composable maps  $f$  and  $g$  we have  $F(gf) = F(g)F(f)$ . These are called *strict homomorphisms*. In real life, these strict homomorphisms are rare; much more common are *homomorphisms*, which preserve composition and identities only up to isomorphisms, such as  $\varphi_{g,f} : F(g)F(f) \cong F(gf)$  and  $\varphi_A^0 : 1_{FA} \cong F1_A$ , which are themselves required to satisfy certain coherence conditions. When the  $\varphi_A^0$  are identities, so that the identities at least are preserved strictly, a homomorphism is said to be *normal*. There is also a still more general notion, called a *lax functor*, which involves comparisons  $\varphi_{g,f} : F(g)F(f) \rightarrow F(gf)$  and  $\varphi_A^0 : 1_{FA} \rightarrow F1_A$  which still satisfy the coherence conditions, but are no longer required to be invertible.

There is a category of bicategories and lax functors, and this has various subcategories in which the morphisms are restricted to the homomorphisms, the normal homomorphisms, or the strict homomorphisms, as the case may be. Just as in the one-dimensional case, one could stop here, and consider the category of bicategories and homomorphisms, or one of its variants, and for some purposes this suffices. But just as one gains a deeper understanding of categories by regarding them as objects of a 2-category, so one gains a deeper understanding of bicategories by regarding them as the objects of a 3-dimensional structure called a tricategory [4].

Just as functoriality can be weakened, so can naturality, giving rise to structures such as oplax natural transformations and pseudonatural transformations, whose precise definitions are given in the following section. These transformations serve as 2-cells, but there are also 3-cells between them, called modifications. The most important 3-dimensional category of bicategories is the tricategory **Hom** [19, 4], mentioned in the introduction, which consists of the following levels of structure:

- bicategories as 0-cells
- homomorphisms between bicategories as 1-cells
- pseudonatural transformations between homomorphisms as 2-cells
- modifications between pseudonatural transformations as 3-cells.

Now the 2-cells of primary interest in this paper, called icons, are not in general pseudonatural transformations. They are obtained by first generalizing

from pseudonatural transformations to oplax natural transformations, but then specializing to those whose components are identities, and so might perhaps be called *Identity Component Oplax Natural* transformations, whence the name *icon*.

As well as the various reasons for considering icons which were given in the introduction, we can now add:

**They are the “costrict” transformations.** Among all the oplax natural transformations, one can identify the 2-natural transformations abstractly via a notion of strictness. The dual notion of “costrictness” identifies exactly the icons.

Formally, the notion of strictness is defined in a 2-dimensional structure called a *sesquicategory* [20]. A sesquicategory  $\mathcal{K}$  consists of an underlying category  $\mathcal{K}_1$ , with a functor  $\mathcal{K} : \mathcal{K}_1^{\text{op}} \times \mathcal{K}_1 \rightarrow \mathbf{Cat}_1$  whose composite with the set-of-objects functor  $\mathbf{Cat}_1 \rightarrow \mathbf{Set}$  is just the hom-functor  $\mathcal{K}_1^{\text{op}} \times \mathcal{K}_1 \rightarrow \mathbf{Set}$ . (There is also an alternative definition: sesquicategories are categories enriched in  $\mathbf{Cat}_1$  with the “funny tensor product”: see [20].)

Spelling out the definition a little, there are objects  $A, B, C, \dots$ ; 1-cells like  $f : A \rightarrow B$ , and 2-cells  $\alpha : f \rightarrow g : A \rightarrow B$ . There is a strictly associative and unital composition of 1-cells, and a strictly associative and unital *vertical* composition of 2-cells (allowing the composite of  $\alpha : f \rightarrow g$  and  $\beta : g \rightarrow h$ ), and 2-cells can be composed on either side with 1-cells, and this is also associative and unital. On the other hand there is no specified *horizontal* composite of 2-cells, as in

$$\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B} \begin{array}{c} \xrightarrow{H} \\ \Downarrow \beta \\ \xrightarrow{K} \end{array} \mathcal{C}.$$

Now we *can* form  $\beta G$  and  $H\alpha$ , and then compose these, or alternatively we can form  $K\alpha$  and  $\beta F$  and compose them, and in each case we obtain a 2-cell from  $HF$  to  $KG$ , but there is no need for the two resulting 2-cells to be equal, as in

$$\begin{array}{ccc} \mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B} \begin{array}{c} \xrightarrow{H} \\ \Downarrow \beta \\ \xrightarrow{K} \end{array} \mathcal{C} & & \mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B} \begin{array}{c} \xrightarrow{H} \\ \Downarrow \beta \\ \xrightarrow{K} \end{array} \mathcal{C} \\ \mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B} \begin{array}{c} \xrightarrow{H} \\ \Downarrow \beta \\ \xrightarrow{K} \end{array} \mathcal{C} & = & \mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B} \begin{array}{c} \xrightarrow{H} \\ \Downarrow \beta \\ \xrightarrow{K} \end{array} \mathcal{C} \end{array}$$

This equation is sometimes called the “middle-four interchange” law. A 2-category is precisely a sesquicategory in which this middle-four interchange law holds for all  $\alpha$  and  $\beta$ .

**Definition 2.1** A 2-cell  $\beta$  in a sesquicategory is *strict* if the equation above holds for all  $\alpha$ . Dually,  $\alpha$  is *costrict* if the equation holds for all  $\beta$ .

### 3 The definition

Bicategories and lax functors form a category: in particular, composition of lax functors is strictly associative. There is also a subcategory consisting only of the homomorphisms. Our goal is to make both of these into 2-categories by introducing 2-cells.

The associativity and identity isomorphisms  $h(gf) \cong h(gf)$  and  $f1 \cong 1f$  in a bicategory will rarely be mentioned explicitly, and will not be given a particular name. A lax functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  involves *lax functoriality constraints*  $\varphi_{g,f} : Fg.Ff \rightarrow F(gf)$  and *identity constraints*  $\varphi_A^0 : 1_{FA} \rightarrow F1_A$ , usually abbreviated to  $\varphi$  and  $\varphi^0$ . In the case of a lax functor called  $G$  we use  $\psi$  and  $\psi^0$ .

Consider bicategories  $\mathcal{A}$  and  $\mathcal{B}$ , and lax functors  $F, G : \mathcal{A} \rightarrow \mathcal{B}$ . The data for an *oplax natural transformation* from  $F$  to  $G$  consists of a 1-cell  $\alpha A : FA \rightarrow GA$  in  $\mathcal{B}$  for every object  $A \in \mathcal{A}$ , and a 2-cell

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \alpha A \downarrow & \Downarrow \alpha f & \downarrow \alpha B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

in  $\mathcal{B}$  for every 1-cell  $f : A \rightarrow B$  in  $\mathcal{A}$ . These are subject to three conditions.

(ON0)  $\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \alpha A \downarrow & \Downarrow \alpha f & \downarrow \alpha B \\ GA & \xrightarrow{Gf} & GB \end{array} = \begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \alpha A \downarrow & \Downarrow F\rho & \downarrow \alpha B \\ GA & \xrightarrow{Gg} & GB \end{array}$

for all 2-cells  $\rho : f \rightarrow g$  in  $\mathcal{A}$

$$\begin{array}{ccc}
\text{(ON1)} & \begin{array}{ccccc}
FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC \\
\alpha A \downarrow & \Downarrow \alpha f & \downarrow \alpha B & \Downarrow \alpha g & \downarrow \alpha C \\
GA & \xrightarrow{Gf} & GB & \xrightarrow{Gg} & GC \\
& \searrow \psi_{f,g} & \swarrow & & \\
& G(gf) & & & 
\end{array} & = & \begin{array}{ccccc}
FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC \\
\alpha A \downarrow & \searrow \varphi_{f,g} & \swarrow & & \downarrow \alpha C \\
& F(gf) & & & \\
& \Downarrow \alpha(gf) & & & \\
GA & & & & GC \\
& \searrow & \swarrow & & \\
& G(gf) & & & 
\end{array}
\end{array}$$

for all composable 1-cells  $f : A \rightarrow B$  and  $g : B \rightarrow C$  in  $\mathcal{A}$

$$\begin{array}{ccc}
\text{(ON2)} & \begin{array}{ccc}
FA & \xrightarrow{1_{FA}} & FA \\
\alpha A \downarrow & \Downarrow & \downarrow \alpha A \\
GA & \xrightarrow{1_{GA}} & GA \\
& \searrow \psi_A^0 & \swarrow \\
& G1_A & 
\end{array} & = & \begin{array}{ccc}
FA & \xrightarrow{1_{FA}} & FA \\
\alpha A \downarrow & \searrow \varphi_A^0 & \swarrow \alpha A \\
& F1_A & \\
& \Downarrow \alpha 1_A & \\
GA & \xrightarrow{Gg} & GA \\
& \searrow & \swarrow \\
& Gg & 
\end{array}
\end{array}$$

for all  $A \in \mathcal{A}$ , where the unnamed 2-cell is the composite of the canonical isomorphisms  $(\alpha A)1_{FA} \cong \alpha A \cong 1_{GA}(\alpha A)$ .

Here the  $\alpha A$  are called the *components* of the transformation, and the  $\alpha f$  are called the *oplax naturality constraints*.<sup>1</sup> The oplax natural transformation  $\alpha$  is *strict* if the  $\alpha f$  are all identity 2-cells. The case where all the  $\alpha f$  are invertible is important too: then  $\alpha$  is said to be a *pseudonatural* transformation. But our main interest will be in the case where the *components* are identity 1-cells.

Now the bicategories, lax functors, and oplax natural transformations do *not* form a 2-category. Restricting from bicategories to 2-categories and from lax functors to 2-functors would solve two of the problems listed below, but not the third.

### First problem

This involves the vertical composition of 2-cells. Given oplax natural transformations  $\alpha : F \rightarrow G$  and  $\beta : G \rightarrow H$ , the composite  $\beta\alpha : F \rightarrow H$  has component at  $A$  given by the composite  $\beta A.\alpha A : FA \rightarrow HA$ . Since composition in  $\mathcal{B}$  is not required to be strictly associative, there is no reason why composition of oplax natural transformations should be strictly associative. Notice that it's no good saying that we'll just try to get a bicategory of bicategories instead: composition of 1-cells *is* associative, it is the composition

<sup>1</sup>There are good reasons for regarding both the  $\alpha A$  and the  $\alpha f$  as components of different type, but this point of view will not be used here.

of *2-cells* which is the problem, and in a 2-category or bicategory there is no room for weakening that; that would require moving to a 3-dimensional structure such as a tricategory.

However this first problem could be solved by requiring the objects of our desired 2-dimensional category to be 2-categories rather than general bicategories. If  $\mathcal{A}$  and  $\mathcal{B}$  are 2-categories, then there *is* a category  $\mathbf{Oplax}_{2l}(\mathcal{A}, \mathcal{B})$  of lax functors from  $\mathcal{A}$  to  $\mathcal{B}$  and oplax natural transformations, so we might hope that it is the hom-category for a putative 2-category or bicategory of bicategories.

### Second problem

The second problem arises when we try to define composition functors  $\mathbf{Oplax}_{2l}(\mathcal{B}, \mathcal{C}) \times \mathbf{Oplax}_{2l}(\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{Oplax}_{2l}(\mathcal{A}, \mathcal{C})$ . On objects we can just use composition of lax functors, with which there is no problem. Given  $\alpha : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$  and  $H : \mathcal{B} \rightarrow \mathcal{C}$  we need to be able to define  $H\alpha : HF \rightarrow HG$ . This would involve components  $H\alpha A : HFA \rightarrow HGA$  and the obvious choice is simply to apply  $H$  to  $\alpha A : FA \rightarrow GA$ . We now need oplax naturality maps

$$\begin{array}{ccc} HFA & \xrightarrow{H\alpha A} & HGA \\ HFf \downarrow & \xRightarrow{(H\alpha)f} & \downarrow HGf \\ HFB & \xrightarrow{H\alpha B} & HGB \end{array}$$

but here we cannot simply apply  $H$  to  $\alpha f : \alpha B.Ff \rightarrow Gf.\alpha A$ , since this would give a map  $H(\alpha B.Ff) \rightarrow H(Gf.\alpha A)$ . We could use the lax functoriality constraint of  $H$  to get a map  $H\alpha B.HFf \rightarrow H(Gf.\alpha A)$ , but the codomain would still be wrong. We could fix this in the same way if  $H$  were not just a lax functor but a homomorphism, but problems would still remain.

### Third problem

Even if we restricted to 2-categories and 2-functors there would still be a problem, once again involving the definition of composition. Given  $\alpha : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$  and  $\beta : H \rightarrow K : \mathcal{B} \rightarrow \mathcal{C}$  we can define  $H\alpha$  and  $\beta F$  (and  $K\alpha$ , and  $\beta G$ ) and so we do obtain a sesquicategory  $\mathbf{Oplax}_2$ . But the middle four



interchange law, asserting the equality of the vertical composites

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & F & & H & \\
 \mathcal{A} & \xrightarrow{\quad} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{C} \\
 & \Downarrow \alpha & & & \\
 & G & & H & \\
 \mathcal{A} & \xrightarrow{\quad} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{C} \\
 & & & \Downarrow \beta & \\
 & & & K & 
 \end{array}
 & = &
 \begin{array}{ccccc}
 & F & & H & \\
 \mathcal{A} & \xrightarrow{\quad} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{C} \\
 & & & \Downarrow \beta & \\
 & & & K & \\
 & F & & & \\
 \mathcal{A} & \xrightarrow{\quad} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{C} \\
 & \Downarrow \alpha & & & \\
 & G & & & \\
 & & & K & 
 \end{array}
 \end{array}$$

need not hold. It does hold when  $\beta$  is 2-natural (strictly natural) but not in general; and there is once again no room to weaken this law: that would require a third dimension.

### Solution

There is, however, another way of restricting the 2-cells which solves all three problems. We still allow arbitrary bicategories as objects, and arbitrary lax functors as 1-cells, but an oplax natural transformation  $\alpha : F \rightarrow G$  is allowed as a 2-cell only if the components  $\alpha_A$  are all identity 1-cells. Such an  $\alpha$  will be called an *icon*.

If the components of  $\alpha$  are to be identities, then  $F$  and  $G$  have to agree on objects. In fact we then omit the components altogether, so as to avoid any problems with the identities in  $\mathcal{B}$  being non-strict. We shall now formulate the precise definition. We use  $M$  to denote a composition map such as  $\mathcal{A}(B, C) \times \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, C)$ , except that to save space we shall write this as  $M_{A,B,C} : \mathcal{A}_2 \rightarrow \mathcal{A}_1$ . Similarly we write  $j_A : 1 \rightarrow \mathcal{A}_1$  for the map  $1 \rightarrow \mathcal{A}(A, A)$  picking out the identity on  $A$ . Recall also that we write  $\varphi$  and  $\varphi^0$  for the lax-functoriality constraints of a lax functor  $F$ , and similarly  $\psi$  and  $\psi^0$  for  $G$ .

**Definition 3.1** If  $F, G : \mathcal{A} \rightarrow \mathcal{B}$  are lax functors of bicategories, and  $FA = GA$  for all objects  $A \in \mathcal{A}$ , an *icon* from  $F$  to  $G$  consists of a natural transformation

$$\mathcal{A}(A, B) \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B}(FA, FB)$$

for all objects  $A, B \in \mathcal{A}$ , subject to the following two conditions:

(ICON1)

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{A}_2 & \xrightleftharpoons[F \times F]{\Downarrow \alpha \times \alpha} & \mathcal{B}_2 \\
 \downarrow M_{A,B,C} & \Downarrow \psi & \downarrow M_{FA,FB,FC} \\
 \mathcal{A}_1 & \xrightarrow{G} & \mathcal{B}_1
 \end{array} & = & \begin{array}{ccc}
 \mathcal{A}_2 & \xrightarrow{F \times F} & \mathcal{B}_2 \\
 \downarrow M_{A,B,C} & \Downarrow \varphi & \downarrow M_{FA,FB,FC} \\
 \mathcal{A}_1 & \xrightleftharpoons[G]{F} & \mathcal{B}_1
 \end{array}
 \end{array}$$

(ICON2)

$$\begin{array}{ccc}
 \begin{array}{ccc}
 1 & & \\
 j_A \swarrow & \xleftarrow[\varphi_0]{} & \searrow j_{FA} \\
 \mathcal{A}_1 & \xrightleftharpoons[G]{F} & \mathcal{B}_1
 \end{array} & = & \begin{array}{ccc}
 1 & & \\
 j_A \swarrow & \xleftarrow[\psi_0]{} & \searrow j_{FA} \\
 \mathcal{A}_1 & \xrightarrow{G} & \mathcal{B}_1
 \end{array}
 \end{array}$$

Here (ON0) corresponds to the naturality of  $\alpha : F \rightarrow G : \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ , while (ON1) and (ON2) correspond to (ICON1) and (ICON2), respectively.

There is now no problem in obtaining a 2-category **Bicat**<sub>2</sub> of bicategories, lax functors, and icons. For bicategories  $\mathcal{A}$  and  $\mathcal{B}$ , the hom-category **Bicat**<sub>2</sub>( $\mathcal{A}, \mathcal{B}$ ) has lax functors from  $\mathcal{A}$  to  $\mathcal{B}$  as objects, and icons as morphisms. Given icons  $\alpha : F \rightarrow G$  and  $\beta : G \rightarrow H$ , we have  $FA = GA = HA$  for all  $A \in \mathcal{A}$ , and the composite  $\beta\alpha : F \rightarrow H$  is defined using the composite natural transformations

$$\begin{array}{ccc}
 \mathcal{A}(A, B) & \xrightleftharpoons[G]{F} & \mathcal{B}(FA, FB) \\
 & \Downarrow \alpha & \\
 & \Downarrow \beta & \\
 & \xrightarrow{H} & 
 \end{array}$$

and composition of natural transformations is of course associative, so that we do indeed have a category **Bicat**<sub>2</sub>( $\mathcal{A}, \mathcal{B}$ ).

Given an icon  $\alpha : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$  and lax functors  $H : \mathcal{B} \rightarrow \mathcal{C}$  and  $K : \mathcal{D} \rightarrow \mathcal{A}$ , the composite  $H\alpha K : HF K \rightarrow HG K$  is defined using the natural transformations

$$\mathcal{D}(D, E) \xrightarrow{K} \mathcal{A}(KD, KE) \xrightleftharpoons[G]{F} \mathcal{B}(FKD, FKE) \xrightarrow{H} \mathcal{C}(HF KD, HF KE)$$

and the middle-four interchange law holds, so that we do indeed have a 2-category **Bicat**<sub>2</sub>:

**Theorem 3.2** *There is a 2-category **Bicat**<sub>2</sub> of bicategories, lax functors, and icons, with the usual composition laws.*

## 4 Bicategories as many-object monoidal categories

Every monoidal category  $\mathcal{V}$  determines a one-object bicategory  $\Sigma\mathcal{V}$ . If we write  $*$  for the unique object of  $\Sigma\mathcal{V}$ , then the hom-category  $\Sigma\mathcal{V}(*,*)$  is the category  $\mathcal{V}$ , composition is given by  $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ , and the identity on  $*$  by the unit object  $I$ . A monoidal functor  $\mathcal{V} \rightarrow \mathcal{W}$  is the same thing as a lax functor  $\Sigma\mathcal{V} \rightarrow \Sigma\mathcal{W}$ . But now if  $F, G : \mathcal{V} \rightarrow \mathcal{W}$  are monoidal functors, and  $\Sigma F, \Sigma G : \Sigma\mathcal{V} \rightarrow \Sigma\mathcal{W}$  the corresponding lax functors, what would it be to give an oplax natural transformation from  $\Sigma F$  to  $\Sigma G$ ? Since  $\Sigma\mathcal{V}$  has one object, this would have a single component, given by an object  $W \in \mathcal{W}$ . Oplax naturality would mean giving, for every  $V \in \mathcal{V}$ , a morphism  $W \otimes FV \rightarrow GV \otimes W$  satisfying various conditions. This is rather more general than a monoidal transformation from  $F$  to  $G$  (this fact was particularly significant in [3]): a monoidal transformation is a natural transformation  $\alpha : F \rightarrow G$  which is suitably compatible with the monoidal structures of  $F$  and  $G$ , and there is certainly no role for the object  $W$  appearing in an oplax natural transformation  $\Sigma F \rightarrow \Sigma G$ . But an *icon* from  $\Sigma F$  to  $\Sigma G$  is exactly a monoidal natural transformation from  $F$  to  $G$ .

We conclude:

**Theorem 4.1** *The 2-category **MonCat** of monoidal categories, monoidal functors, and monoidal natural transformations can be identified with the full sub-2-category of **Bicat**<sub>2</sub> consisting of the one-object bicategories.*

There is also a corresponding result where we use homomorphisms in place of lax functors and strong monoidal functors in place of monoidal functors.

In some contexts the objects of a bicategory are the most important level of structure. But in other contexts they are really just a way of parametrizing the various 1-cells and 2-cells. The bicategory **Mod** of rings, (bi)modules, and module homomorphisms is a good example of the latter situation. A morphism from  $R$  to  $S$  is a left  $R$ -, right  $S$ -bimodule, and a 2-cell is a homomorphism of such bimodules. Much of the time the rings are of rather secondary importance. In situations like this, one can think of a bicategory

as a “many-object” monoidal category. Our 2-category **Bicat**<sub>2</sub> is particularly well-suited to dealing with bicategories thought of in this light. In the remainder of this section we look at equivalences in **Bicat**<sub>2</sub> as an example of the sort of thing we have in mind.

In any 2-category one has the notion of equivalence: a 1-cell  $f : A \rightarrow B$  with a  $g : B \rightarrow A$  satisfying  $gf \cong 1$  and  $fg \cong 1$ . In particular, one could apply this to the case of **Bicat**<sub>2</sub>. First we characterize the invertible 2-cells:

**Proposition 4.2** *An icon  $\alpha$  from  $F$  to  $G$  is invertible if and only if for all objects  $A$  and  $B$  the natural transformation*

$$\mathcal{A}(A, B) \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B}(FA, FB)$$

*is invertible. This in turn is saying that the oplax natural transformation is pseudonatural.*

**Proposition 4.3** *A lax functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is an equivalence in **Bicat**<sub>2</sub> if and only if (i)  $F$  is bijective on objects, (ii) each  $F_{A,B} : \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$  is an equivalence of categories, and (iii)  $F$  is a homomorphism.*

PROOF: An equivalence in **Bicat**<sub>2</sub> consists of a lax functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  for which there is another lax functor  $G : \mathcal{B} \rightarrow \mathcal{A}$  with invertible icons  $GF \cong 1$  and  $FG \cong 1$ . The existence of these icons already forces the object-parts of  $F$  and  $G$  to be mutually inverse. In order to analyze the remaining structure of an equivalence in **Bicat**<sub>2</sub>, we may as well suppose that  $\mathcal{A}$  and  $\mathcal{B}$  have the same objects, and that  $F$  and  $G$  act as the identity on objects. For each pair  $A, B \in \mathcal{A}$  of objects, we then have functors  $F_{A,B} : \mathcal{A}(A, B) \rightarrow \mathcal{B}(A, B)$  and  $G_{A,B} : \mathcal{B}(A, B) \rightarrow \mathcal{A}(A, B)$ , and the invertible icons provide natural isomorphisms  $G_{A,B}F_{A,B} \cong 1$  and  $F_{A,B}G_{A,B} \cong 1$ , thus each  $F_{A,B}$  is indeed an equivalence.

We still need to show that  $F$  is a homomorphism. Consider the invertible icon  $\beta : GF \cong 1$ . For 1-cells  $f : A \rightarrow B$  and  $g : B \rightarrow C$  in  $\mathcal{A}$ , we have the comparisons  $\varphi_{g,f} : Fg.Ff \rightarrow F(gf)$  and  $\psi_{Fg,Ff} : GFg.GFf \rightarrow G(Fg.Ff)$ . Part of the condition of  $\beta$  to be an icon is the commutativity of

$$\begin{array}{ccc} GFg.GFf & \xrightarrow{\psi_{Fg,Ff}} & G(Fg.Ff) \xrightarrow{G\varphi_{g,f}} GF(gf) \\ \beta, \beta \downarrow & & \downarrow \beta \\ gf & \xlongequal{\quad\quad\quad} & gf, \end{array}$$

in which the vertical maps are invertible, and so the composite across the top is so too. Thus  $\psi_{Fg,Ff}$  is split monic for all  $f$  and  $g$ ; since  $F$  is an equivalence on hom-categories, it follows that  $\psi_{h,k} : Gh.Gk \rightarrow G(hk)$  is split monic for all  $h : A \rightarrow B$  and  $k : B \rightarrow C$  in  $\mathcal{B}$ . By symmetry, each  $\varphi_{g,f}$  is also split monic. On the other hand,  $G\varphi_{g,f}$  is split epi, and so since  $G$  is an equivalence on hom-categories, each  $\varphi_{g,f}$  is split epi, and so invertible. The argument that each  $\varphi_0^0 : 1_A \rightarrow F1_A$  is invertible is similar but easier, and so we conclude that  $F$  is indeed a homomorphism.

Now we turn to the converse: any lax functor satisfying the three conditions is an equivalence. It is well-known that for any homomorphism of bicategories  $F : \mathcal{A} \rightarrow \mathcal{B}$  which is an equivalence on hom-categories and surjective on objects up to equivalence, there is a homomorphism  $G : \mathcal{B} \rightarrow \mathcal{A}$  with the composites  $GF$  and  $FG$  each pseudonaturally equivalent to the relevant identity homomorphism; such an  $F$  is called a biequivalence. More precisely, given such an  $F : \mathcal{A} \rightarrow \mathcal{B}$  and a choice, for each  $B \in \mathcal{B}$  of an object  $GB \in \mathcal{A}$  and an equivalence  $eB : FGB \rightarrow B$ , one can extend  $G$  to a homomorphism of bicategories, so that the  $eB$  become the components of a pseudonatural transformation. Furthermore, since  $F$  is an equivalence on hom-categories, we can find for each  $A \in \mathcal{A}$  a morphism  $dA : GFA \rightarrow A$  in  $\mathcal{A}$  with  $FdA : FGFA \rightarrow FA$  isomorphic to  $eFA : FGFA \rightarrow FA$ , and any such choice can be extended to a pseudonatural equivalence  $d : GF \rightarrow 1$ .

In our case,  $F$  is bijective on objects, so we can choose  $GB$  so that  $FGB = B$ . Thus we obtain a homomorphism  $G : \mathcal{B} \rightarrow \mathcal{A}$  and an invertible icon  $FG \cong 1$ . Now the  $eB$  of the previous paragraph is the identity, so we can take  $dA : GFA \rightarrow A$  also to be the identity, and then the resulting  $d : GF \rightarrow 1$  becomes our invertible icon  $GF \cong 1$ .  $\square$

It is really the biequivalences that provide the general notion of “sameness” for bicategories; for example Gabriel-Ulmer duality asserts the biequivalence, in this sense, of the 2-category **Lex** of categories with finite limits, and a certain 2-category **LFP** of locally finitely presentable categories. This biequivalence involves genuine content at the object-level, and is certainly not an equivalence in **Bicat**<sub>2</sub>.

On the other hand, there are some important examples, where “nothing happens” with the objects. A good example is the theorem that every bicategory is biequivalent to a 2-category (“every bicategory can be made strict”) [15]. Here the lack of strictness has nothing to do with the objects, and they can be left unchanged; the problem is rather with the 1-cells. Thus it is the case that every bicategory is equivalent in **Bicat**<sub>2</sub> to a strict one, and we can conclude (see also [13]):

**Theorem 4.4** *The 2-category  $\mathbf{Bicat}_2$  is biequivalent to the full sub-2-category consisting of the strict bicategories (the 2-categories).*

## 5 Costrictness

In this section we describe an abstract characterization of the icons among all oplax natural transformations. We restrict, for convenience, to the case of 2-categories and 2-functors, although a similar analysis could be given involving bicategories and homomorphisms (not lax functors). The formal notion of strictness in a sesquicategory was defined in Section 2; here we work through the idea more gently in the specific case of the sesquicategory  $\mathbf{Oplax}_2$  of 2-categories, 2-functors, and oplax natural transformations.

Consider then a diagram of 2-categories, 2-functors, and oplax natural transformations

$$\begin{array}{ccc} & F & \\ \mathcal{A} & \xrightarrow{\quad} & \mathcal{B} \\ & \Downarrow \alpha & \\ & G & \end{array} \quad \begin{array}{ccc} & H & \\ \mathcal{B} & \xrightarrow{\quad} & \mathcal{C} \\ & \Downarrow \beta & \\ & K & \end{array}.$$

As described in Section 2 above, we can form  $\beta G : HG \rightarrow KG$  and  $H\alpha : HF \rightarrow HG$ , and their composite  $\beta G.H\alpha : HF \rightarrow KG$ ; and similarly the composite  $K\alpha.\beta F : HF \rightarrow KG$ . The middle-four interchange law would require these composites to be the same, but in general they are not.

We can nonetheless, consider situations under which they are the same. We start with the following question:

For which  $\beta : H \rightarrow K : \mathcal{B} \rightarrow \mathcal{C}$  is it the case that middle-four interchange holds for *all*  $\alpha : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$ ?

In particular, it would have to hold when  $\mathcal{A}$  is the terminal 2-category  $\mathbf{1}$ . Then  $F$  and  $G$  are just objects of  $\mathcal{B}$ , and  $\alpha : F \rightarrow G$  a morphism, and middle-four interchange says precisely that the components  $\beta B : HB \rightarrow KB$  are strictly natural. Notice that this does not yet imply that  $\beta$  is itself strictly natural, for we might have strictly natural components yet still choose non-trivial oplax naturality maps.

But now let  $\mathcal{A}$  be the arrow 2-category  $\mathbf{2}$ . A 2-functor  $\mathbf{2} \rightarrow \mathcal{B}$  is just a morphism  $f : A \rightarrow B$  in  $\mathcal{B}$ . An oplax natural transformation between two

such 2-functors  $\mathbf{2} \rightarrow \mathcal{B}$ , say  $f : A \rightarrow B$  and  $g : C \rightarrow D$ , is a square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \downarrow & \Downarrow \varphi & \downarrow v \\ C & \xrightarrow{g} & D. \end{array}$$

Now let  $G : \mathbf{2} \rightarrow \mathcal{B}$  correspond to a map  $f : A \rightarrow B$  in  $\mathcal{B}$ , let  $F : \mathbf{2} \rightarrow \mathcal{B}$  correspond to  $1 : A \rightarrow A$ , and let  $\alpha : F \rightarrow G$  correspond to the strict oplax natural transformation with components  $1 : A \rightarrow A$  and  $f : A \rightarrow B$ , as in

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \parallel & & \downarrow f \\ A & \xrightarrow{f} & B. \end{array}$$

Then the equation  $\beta G.H\alpha = K\alpha.\beta F$  becomes

$$\begin{array}{ccc} HA & \xrightarrow{1} & HA \\ 1 \downarrow & & \downarrow Hf \\ HA & \xrightarrow{Hf} & HB \\ \parallel & & \parallel \\ KA & \xrightarrow{Kf} & KB \end{array} \quad = \quad \begin{array}{ccc} HA & \xrightarrow{1} & HA \\ \parallel & \Downarrow \beta 1_A & \parallel \\ KA & \xrightarrow{1} & KA \\ 1 \downarrow & & \downarrow Kf \\ KA & \xrightarrow{Kf} & KB \end{array}$$

but  $\beta 1_A$  is the identity, and so this says that  $\beta f$  is the identity. Since  $f$  was arbitrary, this says that  $\beta$  really is strict.

Conversely, if  $\beta$  is strict, then clearly the middle-four interchange law will hold for all  $\alpha$ . But now we have an internal (representable) notion of strictness, and so we can consider the dual:

For which  $\alpha : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$  is it the case that middle-four interchange holds for *all*  $\beta : H \rightarrow K : \mathcal{B} \rightarrow \mathcal{C}$ ?

Formally, one could define strictness in any sesquicategory. Then our original notion would be strictness in **Oplax**<sub>2</sub>, while the question just posed asks which  $\alpha$  are strict in **Oplax**<sub>2</sub><sup>op</sup>. Such an  $\alpha$  will be called *costrict*. We shall show that  $\alpha$  is a costrict if and only if it is an icon.

Let  $\alpha$  and  $\beta$  be as above, with  $\alpha$  an icon. We must show that  $\beta G.H\alpha = K\alpha.\beta F$ , but this amounts to showing that these composites have the same component for each  $A \in \mathcal{A}$ , and the same oplax naturality map for each  $f : A \rightarrow B$  in  $\mathcal{A}$ . Now  $FA = GA$  and  $\alpha A$  is the identity, so  $(\beta G.H\alpha)A = \beta GA.H\alpha A = \beta GA = \beta FA = K\alpha A.\beta FA = (K\alpha.\beta F)A$  and the components agree. As for the second part, it asserts the equality of pasting composites

$$\begin{array}{ccc}
\begin{array}{ccc}
HFA & \xrightarrow{HFf} & HFB \\
\parallel & \Downarrow H\alpha f & \parallel \\
HGA & \xrightarrow{HGf} & HGB \\
\downarrow \beta GA & \Downarrow \beta Gf & \downarrow \beta GB \\
KGA & \xrightarrow{KGf} & KGB
\end{array} & = & 
\begin{array}{ccc}
HFA & \xrightarrow{HFf} & HFB \\
\downarrow \beta FA & \Downarrow \beta Ff & \downarrow \beta FB \\
KFA & \xrightarrow{KFf} & KFB \\
\parallel & \Downarrow K\alpha f & \parallel \\
KGA & \xrightarrow{KGf} & KGB
\end{array}
\end{array}$$

but this follows by (ON0) for the oplax natural transformation  $\beta$ .

Suppose conversely, that  $\alpha$  is strict. We shall show that it must be an icon. We do this by constructing a 2-category  $\mathcal{C}$  with 2-functors  $H, K : \mathcal{B} \rightarrow \mathcal{C}$  and an oplax natural transformation  $\beta : H \rightarrow K$  with the property that if  $g : B \rightarrow C$  is a morphism in  $\mathcal{B}$  with respect to which  $\beta$  is strictly natural, then  $g$  must be an identity. Since  $\beta$  is by assumption strictly natural with respect to each component  $\alpha A : FA \rightarrow GA$ , it will follow that these components are identities, and so that  $\alpha$  is an icon.

How do we construct such a  $\mathcal{C}$ ? Among all the 2-categories  $\mathcal{C}$  equipped with 2-functors  $H, K : \mathcal{B} \rightarrow \mathcal{C}$  and an oplax natural transformation  $\beta : H \rightarrow K$ , there is a universal one, given by the (lax) Gray tensor product  $\mathbf{2} \otimes \mathcal{B}$  of the arrow category  $\mathbf{2} = \{0 < 1\}$  and  $\mathcal{B}$ . The objects of  $\mathbf{2} \otimes \mathcal{B}$  are pairs  $(i, B)$  where  $i \in \mathbf{2}$  and  $B \in \mathcal{B}$ . The morphisms of  $\mathbf{2} \otimes \mathcal{B}$  are freely generated by  $(!, B) : (0, B) \rightarrow (1, B)$  for each  $B \in \mathcal{B}$ , and  $(i, g) : (i, B) \rightarrow (i, C)$  for each  $i \in \mathbf{2}$  and  $g : B \rightarrow C$  in  $\mathcal{B}$ ; subject to the relations that  $(i, hg) = (i, h)(i, g)$  and  $(i, 1_B) = 1_{(i, B)}$ . The 2-cells are generated by  $(i, \beta) : (i, g) \rightarrow (i, h)$  for



all 2-cells  $\beta : g \rightarrow h$  in  $\mathcal{B}$ , and

$$\begin{array}{ccc} (0, B) & \xrightarrow{(0, g)} & (0, C) \\ (!, B) \downarrow & \Downarrow (!, g) & \downarrow (!, C) \\ (1, B) & \xrightarrow{(1, g)} & (1, C) \end{array}$$

for all 1-cells  $g : B \rightarrow C$ , subject to certain relations. Which ones? The 2-functors  $H$  and  $K$  are defined so as to send  $B \in \mathcal{B}$  to  $(0, B)$  and to  $(1, B)$ , respectively, and similarly on morphisms and 2-cells. We want the  $(!, B) : (0, B) \rightarrow (0, B)$  to be the components  $HB \rightarrow KB$  of an oplax natural transformation, with oplax naturality constraints  $(!, g)$ . The relations are exactly the ones which make this work. (See [6] or [20] for details.) The important thing for us is that  $(!, g)$  is not an identity 2-cell unless  $g$  is an identity 1-cell, but in fact we can see this without knowing exactly what the 2-cells are, because the domain and codomain 1-cells  $(!, C)(0, g)$  and  $(1, g)(!, B)$  of  $(!, g)$  are different unless  $g$  is an identity 1-cell.

This now proves:

**Theorem 5.1** *An oplax natural transformation between 2-functors is costrict if and only if it is an icon.*

**Remark 5.2** We have defined the notion of strictness in any sesquicategory. Just as one-object 2-categories can be identified with strict monoidal categories, one-object sesquicategories can be identified with structures called *premonoidal categories* [18]. A morphism in a premonoidal category is called *central* [18] if the corresponding 2-cell is both strict and costrict. In the sesquicategory of 2-categories, 2-functors, and oplax natural transformations, the only 2-cells which are strict and costrict are the identities, but in general it is possible to have non-identity central morphisms; see [18].

## 6 Alternative approaches

In this section we give two alternative approaches to icons, one using double categories, the other using 2-dimensional universal algebra.

### 6.1 Bicategories as pseudo double categories

A double category involves two categories with the same set of objects, and with the morphisms called vertical and horizontal, respectively. There

are also “squares” which have a vertical domain, a vertical codomain, a horizontal domain, and a horizontal codomain; and which can be composed either horizontally or vertically. For instance in

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ v \downarrow & \alpha & \downarrow v' \\ C & \xrightarrow{h'} & D \end{array}$$

$h : A \rightarrow B$  and  $h' : C \rightarrow D$  are horizontal arrows, and are the horizontal domain and horizontal codomain of the square  $\alpha$ , while  $v : A \rightarrow C$  and  $v' : B \rightarrow D$  are vertical arrows, and are the vertical domain and vertical codomain of the square.

A 2-category can be thought of as a double category with no non-identity horizontal arrows, so that all “squares” collapse to the usual shape of a 2-cell in a 2-category, as in

$$\begin{array}{c} A \\ h \left( \begin{array}{c} \alpha \\ \Rightarrow \end{array} \right) h' \\ B \end{array}$$

A pseudo double category is a slight generalization of a double category, in which the vertical, but not the horizontal, structure is weakened. A pseudo double category with no non-identity horizontal cells is the same thing as a bicategory.

In [5], Grandis and Paré defined a 2-category **LaxDbI** whose objects were pseudo double categories, whose morphisms are lax functors, and “horizontal transformations” as 2-cells. Here a horizontal transformation involves components which are horizontal morphisms; these are only weakly natural. If we restrict the objects of **LaxDbI** to the bicategories (seen as pseudo double categories with no non-identity horizontal cells), then such horizontal transformations are forced to have identity components, and in fact turn out to be icons.

## 6.2 Bicategories as algebras for a 2-monad

In [13] an alternative approach to icons was also given. There a 2-category **Cat-Gph<sub>2</sub>** of **Cat**-enriched graphs was defined. A **Cat**-graph  $\mathcal{G}$  has vertices  $A, B, C, \dots$ , and “hom-categories”  $\mathcal{G}(A, B)$  for all vertices  $A$  and  $B$ . A homomorphism  $F : \mathcal{G} \rightarrow \mathcal{H}$  of **Cat**-graphs consists of a function  $A \mapsto FA$  sending vertices of  $\mathcal{G}$  to vertices of  $\mathcal{H}$ , and a functor  $F : \mathcal{G}(A, B) \rightarrow \mathcal{H}(FA, FB)$

for all vertices  $A$  and  $B$ . The **Cat**-graphs and their homomorphisms form a category [21], but it is also possible to make it into a 2-category. A 2-cell  $F \rightarrow F'$  is possible only if  $F$  and  $F'$  agree on vertices; it then consists of a natural transformation

$$\mathcal{G}(A,B) \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{F'} \end{array} \mathcal{H}(FA,FB)$$

for all vertices  $A$  and  $B$ , with no further conditions.

This 2-category is locally finitely presentable in the sense of [9], and one can give a presentation, in the sense of [10], of a finitary 2-monad on **Cat-Gph**<sub>2</sub> whose algebras are the bicategories, whose strict morphisms are the strict homomorphisms, and whose algebra 2-cells are the icons. The pseudomorphisms of algebras are the homomorphisms, and so the 2-category **Hom**<sub>2</sub> is the 2-category  $T\text{-}\mathbf{Alg}$  of strict algebras, pseudomorphisms, and algebra 2-cells, which is the main object of study in [2]. It follows that **Hom**<sub>2</sub> has bicategorical limits and colimits, and strict products, inserters, and equifiers. All this was done in [13].

It is not hard to verify similarly that the lax morphisms of algebras are the lax functors. Thus the 2-category **Bicat**<sub>2</sub> of bicategories, lax functors, and icons is the 2-category  $T\text{-}\mathbf{Alg}_1$  of strict algebras, lax morphisms, and algebra 2-cells, which is the main object of study of [12], and so by [12] **Bicat**<sub>2</sub> has oplax limits, and in particular products, cotensor products, Eilenberg-Moore objects of comonads, *some* inserters and equifiers, and any limit of strict morphisms.

## 7 Applications

### 7.1 2-nerves

There is a functor from the category **Cat**<sub>1</sub> of categories to the category  $[\Delta^{\text{op}}, \mathbf{Set}]$  of simplicial sets, which sends a category to its *nerve*. Concretely, the 0-simplices of the nerve are the objects, the 1-simplices the morphisms, the 2-simplices the composable pairs, and so on. But the slick way to define this is to regard  $\Delta$  as the full subcategory of **Cat**<sub>1</sub> consisting of the finite ordinals (the finite totally ordered sets, regarded as categories), and then use the inclusion  $J : \Delta \rightarrow \mathbf{Cat}_1$  to define the nerve  $NC$  of a category  $C$  as the simplicial set **Cat**<sub>1</sub>( $J-, C$ ) sending an object  $\mathbf{n} \in \Delta$  to the set **Cat**<sub>1</sub>( $J\mathbf{n}, C$ ) of functors from  $J\mathbf{n}$  to  $C$ .

More generally, if  $\mathcal{C}$  is a category enriched in  $\mathcal{V}$ , and  $J : \mathcal{A} \rightarrow \mathcal{C}$  is a  $\mathcal{V}$ -functor, each object  $C$  determines a  $\mathcal{V}$ -functor  $\mathcal{C}(J-, C)$  sending  $A \in \mathcal{A}$

to the hom-object  $\mathcal{C}(JA, C) \in \mathcal{V}$ ; now there is a  $\mathcal{V}$ -functor  $\mathcal{C} \rightarrow [\mathcal{A}^{\text{op}}, \mathcal{V}]$  sending  $C$  to  $\mathcal{C}(J-, C)$ .

We shall apply this in the special case where  $\mathcal{V} = \mathbf{Cat}$ , so a  $\mathcal{V}$ -category is a 2-category. We take  $\mathcal{C}$  to be the 2-category  $\mathbf{nHom}_2$  of bicategories, normal homomorphisms, and icons, and we take  $\mathcal{A}$  to be  $\Delta$ , seen as a 2-category with no non-identity 2-cells. We have the inclusion of  $\Delta$  in  $\mathbf{Cat}_1$ , but then any category can be regarded as a bicategory with no non-identity 2-cells, and so we get an inclusion  $J : \Delta \rightarrow \mathbf{nHom}_2$ , which turns out to be fully faithful. The induced  $\mathcal{V}$ -functor  $\mathbf{nHom}_2 \rightarrow [\Delta^{\text{op}}, \mathbf{Cat}]$  sends a bicategory  $B$  to a simplicial object  $\mathbf{nHom}_2(J-, B)$  in  $\mathbf{Cat}$ , called the *2-nerve* of the bicategory. See [13] for details.

## 7.2 Lawvere theories

A Lawvere theory is a category  $\mathcal{T}$  with finite products, equipped with a functor  $\mathcal{S}^{\text{op}} \rightarrow \mathcal{T}$  from the opposite of the category of finite sets, which is bijective on objects and preserves finite products. In other words,  $\mathcal{T}$  is a category with finite products whose objects have the form  $X^n$  for  $n \in \mathbb{N}$ , and with  $X^m \times X^n = X^{m+n}$ . A morphism of Lawvere theories from  $E : \mathcal{S}^{\text{op}} \rightarrow \mathcal{T}$  to  $E' : \mathcal{S}^{\text{op}} \rightarrow \mathcal{T}'$  is a functor  $M : \mathcal{T} \rightarrow \mathcal{T}'$  with  $ME = E'$ ; then  $M$  necessarily preserves finite products.

There is a 2-dimensional version of Lawvere theory (which is a special case of the more general enriched categorical version [16]). In place of  $\mathcal{S}$  we have the 2-category  $\mathcal{C}$  of finitely presentable categories, and now  $\mathcal{T}$  is required to have *cotensors* by objects of  $\mathcal{C}$  (finite cotensors for short) and a bijective-on-objects finite-cotensor-preserving 2-functor  $E : \mathcal{C}^{\text{op}} \rightarrow \mathcal{T}$ . Once again, a morphism of theories from  $E : \mathcal{C}^{\text{op}} \rightarrow \mathcal{T}$  to  $E' : \mathcal{C}^{\text{op}} \rightarrow \mathcal{T}'$  is a 2-functor  $M : \mathcal{T} \rightarrow \mathcal{T}'$  satisfying  $ME = E'$ , and once again such an  $M$  necessarily preserves finite cotensors. But there is now an ingredient not present in the general enriched version: the category of  $\mathcal{V}$ -enriched Lawvere theories is just a category, there is no enrichment over  $\mathcal{V}$ ; but the category of 2-dimensional Lawvere theories does enrich over  $\mathbf{Cat}$  to give a 2-category. This corresponds precisely to the fact that there is a 2-category of finitary 2-monads on  $\mathbf{Cat}$  (studied in [10] and [11]). Given morphisms  $M, M' : \mathcal{T} \rightarrow \mathcal{T}'$  as above, a 2-cell from  $M$  to  $M'$  is an oplax natural transformation  $\varphi : M \rightarrow M'$  whose restriction along  $E$  is the identity transformation on  $E'$ . This condition implies in particular that  $\varphi$  is an icon, and this means that we do indeed get a 2-category of 2-dimensional Lawvere theories. This is important when one wishes to consider *flexibility* of theories.

Furthermore, the morphisms  $M$  and  $M'$  induce 2-functors  $M, M' : \mathbf{Mod}(\mathcal{T}') \rightarrow$

$\mathbf{Mod}(\mathcal{T})$  between the 2-categories of models, and now composition with  $\varphi$  induces a 2-natural transformation  $M \rightarrow M'$ .

See [14] for more details.

### 7.3 Bundles

We have already seen that the 2-category structure of  $\mathbf{Bicat}_2$  can be used to obtain a notion of equivalence of bicategories. It can also be used to obtain a notion of fibration.

Briefly, if  $p : A \rightarrow B$  is a morphism in a 2-category  $\mathcal{K}$ , we can define what it means for  $p$  to be a fibration. First of all, for a morphism  $a : X \rightarrow A$ , a 2-cell  $\alpha : a' \rightarrow a$  is said to be *p-cartesian* if for each  $c : X \rightarrow A$  and each pair  $(\gamma : pc \rightarrow pa', \delta : c \rightarrow a)$  with  $p\alpha \cdot \gamma = p\delta$  there is a unique  $\bar{\gamma} : c \rightarrow a'$  with  $p\bar{\gamma} = \gamma$  and  $\alpha\bar{\gamma} = \delta$ , as in the diagram below.

$$\begin{array}{ccc} c & & \\ & \searrow \delta & \\ & \bar{\gamma} & \\ & \searrow & \\ a' & \xrightarrow{\alpha} & a \end{array}$$
  

$$\begin{array}{ccc} pc & & \\ & \searrow p\delta & \\ & \gamma & \\ & \searrow & \\ pa' & \xrightarrow{p\alpha} & pa \end{array}$$

Then  $p$  is said to be a *fibration* if (i) for each  $a : X \rightarrow A$ ,  $b : X \rightarrow B$ , and  $\beta : b \rightarrow pa$  there exists a  $p$ -cartesian  $\alpha : a' \rightarrow a$  with  $pa' = b$  and  $p\alpha = \beta$ , and (ii) if  $\alpha : a' \rightarrow a$  is  $p$ -cartesian, then  $\alpha x : a'x \rightarrow ax$  is  $p$ -cartesian for any  $x : Y \rightarrow X$ .

One can now consider fibrations in the 2-category  $\mathbf{Bicat}_2$  of bicategories, normal homomorphisms, and icons. A fibration will be a homomorphism  $P : \mathcal{A} \rightarrow \mathcal{B}$  of bicategories, with for each  $f : A \rightarrow A'$  in  $\mathcal{A}$  and each 2-cell  $\beta : g \rightarrow Pf$  in  $\mathcal{B}$  a 2-cell  $\bar{\beta} : \bar{g} \rightarrow f$  in  $\mathcal{A}$  with  $P\bar{\beta} = \beta$ , as in the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ & \beta \Uparrow & \\ & \bar{\beta} & \\ & \bar{g} & \end{array}$$
  

$$\begin{array}{ccc} PA & \xrightarrow{Pf} & PA' \\ & \beta \Uparrow & \\ & \bar{\beta} & \\ & \bar{g} & \end{array}$$

and with the liftings  $\bar{\beta}$  required to satisfy various properties.

There are various special cases. In particular, one could consider fibration not just in **Bicat**<sub>2</sub>, but in the 2-category **sHom**<sub>2</sub> of bicategories, *strict* homomorphisms, and icons. This imposes further compatibility conditions on the liftings  $\bar{\beta}$ . A still greater restriction would be to ask not just for a fibration in **sHom**<sub>2</sub> but a split fibration. This means that one makes chosen liftings, which are themselves functorial. This situation is being studied in [7] under the name *bundles of bicategories* after it arose in [8]; see [7] for examples and further discussion.

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